Binomial Series

Most of this material is taken from Taylor, "Advanced Calculus". First, recall Theorem 2.55

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x), \text{ where}$$

$$R_n(x) = \frac{x^{n+1}}{n!}\int_0^1 (1-t)^n f^{(n+1)}(tx)\,dt.$$ For the binomial series $f(x) = (1+x)^\alpha$. We will assume α is <u>not</u> a positive integer. Then

$$f^{(n+1)}(x) = \alpha(\alpha-1)\cdots(\alpha-n)(1+x)^{\alpha-n-1}; \quad \frac{f^{n}(0)}{n!} = \binom{\alpha}{n}.$$

By the intermediate value theorem

$$R_n(x) = \frac{1}{n!}(1-\theta_n)^n \alpha(\alpha-1)\cdots(\alpha-n)(1+\theta_n x)^{\alpha-n-1} x^{n+1}$$

Remember θ depends on n, but to simplify notation I'll write $\theta = \theta_n$ and note that $0 < \theta < 1$.

$$R_n(x) = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}\left(\frac{1-\theta}{1+\theta x}\right)^n (1+\theta x)^{\alpha-1} x^{n+1}.$$

$$0 < \frac{1-\theta}{1+\theta x} < 1, \quad \text{since } 1 > 1-\theta > 0, \quad 1+\theta x > 0,$$

$$\iff 1-\theta < 1+\theta x \iff -\theta < \theta x \iff -1 < x,$$

and recall $-1 < x < 1$. If $\alpha > 1$,

$$|1+\theta x|^{\alpha-1} < (1+|x|)^{\alpha-1}; \text{ and if } \alpha < 1$$

$$|1+\theta x|^{\alpha-1} < (1-|x|)^{\alpha-1},$$

Hence $|R_n(x)| \le \frac{|\alpha(\alpha-1)\cdots(\alpha-n)|}{n!}(1 \pm |x|)^{\alpha-1}|x|^{n+1}$

Let S_n denote the right side. Then

$$S_{n+1}/S_n = \left|\frac{\alpha-n-1}{n+1}x\right| \longrightarrow |x| < 1.$$

Hence $S_n \to 0$ so $R_n(x) \to 0$. This proves

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \text{for } |x| < 1.$$

The behavior at $x = \pm 1$ is more delicate. To justify the following statements are a long story (omitted here).

1. $x = -1$.
   (a) The series $\sum \binom{\alpha}{n}(-1)^n$ diverges if $\alpha < 0$ and converges absolutely if $\alpha > 0$.
   (b) If $\alpha > 0$, $(1-1)^\alpha = 0 = \sum_{n=0}^{\infty} \binom{\alpha}{n}(-1)^n$.

   For example $(\alpha = 1/2)$ $\quad 0 = 1 + \frac{1}{2}(-1) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-1)^2 + \frac{1}{6}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-1)^3 + \cdots$

   $$0 = 1 - \frac{1}{2} - \frac{1}{8} - \frac{1}{16} - \frac{5}{128} + \cdots$$

2. $x = 1$.
   (a) The series $\sum \binom{\alpha}{n}$ diverges if $\alpha \le -1$
   (b) If $\alpha > -1$
   $$2^\alpha = \sum \binom{\alpha}{n}$$

For example $(\alpha = 1/2)$, $\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \cdots$